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Infinite Linear Sequential Machines*

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ABSTRACT

Linear sequential machines (LSM's for short) are considered over arbitrary fields. Various finiteness conditions are given for sequential machines and LSM's with these properties are characterized. It is shown that the set of all input/output pairs characterizes an LSM. The effect on realizations of varying the ground field is studied. A number of algorithms are given for the solution of specific problems. Decision procedures are given for the equivalence problem for LSM's. The problem of determining if one state is accessible from another is discussed and a number of results are presented.

INTRODUCTION

In the last ten years, a great deal of research has been concerned with linear sequential machines (LSM's for short). See [6], [11] and [13] for a summary of some of this work. In the present paper, LSM's are considered over arbitrary fields. Most physical systems involve fields such as the rational numbers, the real numbers, or the complex numbers. General systems over a finite field are of interest, but the finiteness assumption makes an LSM a special case of a finite automaton and this special family is not a very important class of finite automata.

Many of the results known about finite LSM's are valid for the case of an arbitrary LSM. It is interesting to note that no decision problems have previously been raised concerning LSM's. Since we consider infinite LSM's, we shall explore certain decision problems. In order to do this, we must describe a field effectively. This leads naturally to the theory of computable fields studied by Rabin [12]. This theory, by necessity, rules out such interesting and important fields as the real numbers or complex numbers. Thus, the theory is very restrictive.

The present paper is divided into six sections. In Section 1, various finiteness con-

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ditions are considered on sequential machines. LSM's satisfying these conditions are characterized. In Section 2, the sequential relations of LSM's are introduced. A proof is given that two LSM's with the same relation are (functionally) equivalent. Section 3 is concerned with linear realizations of sequential machines. The effect of changing fields is studied with respect to realizations. In Section 4, the study of decision problems is begun by showing that it is decidable whether or not two LSM's are equivalent. In Section 5, a variety of decision problems are considered. Section 6 is devoted to the problem of determining whether one state is reachable from another.

The remainder of this introduction gives the formal definitions of sequential machines and LSM's as well as our notational conventions.

DEFINITION. A *sequential machine* is a 5-tuple $M = \langle Q, \Sigma, \Delta, \delta, \lambda \rangle$ where

(i) Q is a nonempty set of *states*.

(ii) Σ is a nonempty set of *input symbols*.

(iii) Δ is a nonempty set of *output symbols*.

(iv) δ is a map from $Q \times \Sigma$ into Q called the *direct transition function*.

(v) λ is a map from $Q \times \Sigma$ into Δ called the *output function*.

We shall say that a sequential machine $M = \langle Q, \Sigma, \Delta, \delta, \lambda \rangle$ is finite when Q, Σ, and Δ are finite.

It is necessary to extend the transition function to a mapping[1] from $Q \times \Sigma^*$ into Q. This is done in the conventional manner as follows.

DEFINITION. Let $M = \langle Q, \Sigma, \Delta, \delta, \lambda \rangle$ be a sequential machine. For each $q \in Q$, $x \in \Sigma^*$, and $a \in \Sigma$.

$$\delta(q, \Lambda) = q,$$
$$\delta(q, xa) = \delta(\delta(q, x), a).$$

There are two common extensions used for the output function λ. It will be necessary for us to use both of them.

DEFINITION. Let $M = \langle Q, \Sigma, \Delta, \delta, \lambda \rangle$ be a sequential machine. For each $q \in Q$, $x \in \Sigma^*$ and $a \in \Sigma$,

$$\lambda(q, \Lambda) = \Lambda,$$
$$\lambda(q, xa) = \lambda(q, x)\, \lambda(\delta(q, x), a),$$

and

$$\hat{\lambda}(q, xa) = \lambda(\delta(q, x), a).$$

[1] If X and Y are sets of words, then the product of X and Y is the set $\{xy \mid x \in X, y \in Y\}$ where xy is the *concatenation* of x and y. For $i \geqslant 1$, write $X^{i+1} = X^i X$ and $X^+ = \cup_{i \geqslant 1} X^i$. Let Λ be the null word and write $X^* = X^+ \cup \{\Lambda\}$. For any word x, $lg(x)$ denotes the *length* of x. Finally, let $\emptyset$ denote the empty set.

Thus $\lambda_q(x) = \lambda(q, x)$ is a length preserving function[2] from Σ^* into Δ^* which is the concatenation of the output symbols produced by the individual input symbols. $\hat{\lambda}_q(x) = \hat{\lambda}(q, x)$ is a map from Σ^+ into Δ which gives the last output symbol produced by M when started in state q and reading input x.

A principal concern of system theorists is the input-output behavior of these machines.

DEFINITION. Let $M = \langle Q, \Sigma, \Delta, \delta, \lambda \rangle$ be a sequential machine. Define

$$F(M) = \{\lambda_q \mid q \in Q\},$$

and

$$R(M) = \{(x, \lambda(q, x)) \mid x \in \Sigma^*, q \in Q\}.$$

$F(M)$ is the *set of functions computed by* M, one function for each internal state. $R(M)$ is the *sequential relation of* M which consists of all input-output pairs of M.

Two sequential machines M_1 and M_2 over the same input and output alphabets are said to be *equivalent* [*relationally equivalent*] if $F(M_1) = F(M_2)$ $[R(M_1) = R(M_2)]$. It is always true that functional equivalence implies relational equivalence, but the converse is false in general [7].

We now introduce the special class of machines with which we shall deal.

If F is a field and m is a nonnegative integer, let F_m be the vector space of column vectors of dimension m over F. Note that $F_0 = \{0\}$.

DEFINITION. A *linear sequential machine* M(LSM for short) is a sequential machine $M = \langle Q, \Sigma, \Delta, \delta, \lambda \rangle$ with the following special properties. There exists a field F and nonnegative integers n, k, and l such that $Q = F_n$, $\Sigma = F_k$, and $\Delta = F_l$. Furthermore there exists an $n \times n$ matrix $\mathbf{A}$, an $n \times k$ matrix $\mathbf{B}$, an $l \times n$ matrix $\mathbf{C}$ and an $l \times k$ matrix $\mathbf{D}$ such that for each $(q, a) \in Q \times \Sigma$

$$\delta(q, a) = \mathbf{A}q + \mathbf{B}a,$$
$$\lambda(q, a) = \mathbf{C}q + \mathbf{D}a.$$

Such a linear sequential machine M will sometimes be denoted by $\langle F, n, k, l, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \rangle$, $\langle F, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \rangle$ or $\langle F, n, k, l, \delta, \lambda \rangle$.

In the preceding definition, we allow $n = 0$. In this case, matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are null and the output function is $\lambda : a \to \mathbf{D}a$ which describes a combinational switching network which is linear. This is the "memoryless" case.

It is possible to deal with LSM's in terms of linear functions and abstract vector spaces rather than matrices and F_n, etc. This leads to shorter proofs, but overlooks certain questions of representability and effectiveness. For this reason, we employ matrices.

[2] A function φ from Σ^* to Δ^* is *length preserving* if $lg(\varphi x) = lg(x)$.

Note that if the field F is finite, then M is finite. In general, most results about LSM's are valid over arbitrary fields. We shall assume F is arbitrary for much of the remainder of this paper.

The following facts are well known[3].

PROPOSITION. *Let* $M = \langle F, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \rangle$ *be an* LSM.

(a) *For each state* $q \in F_n$, *and* $a_0, \ldots, a_{t-1}$ *in* F_k, $t > 0$,

$$\delta(q, a_0 \cdots a_{t-1}) = \mathbf{A}^t q + \sum_{i=0}^{t-1} \mathbf{A}^{t-i-1} \mathbf{B} a_i .$$

(b) $\hat{\lambda}(q, a_0 \cdots a_{t-1}) = \mathbf{C}\mathbf{A}^{t-1} q + \sum_{i=0}^{t-2} \mathbf{C}\mathbf{A}^{t-i-2} \mathbf{B} a_i + \mathbf{D} a_{t-1}$

For $q, q' \in F_n$, x *in* F_k^+, c *in* F.

(c) $\delta(q + cq', x) = \delta(q, x) + c\delta(q', 0^{lg(x)})$.[4]

(d) $\hat{\lambda}(q + cq', x) = \hat{\lambda}(q, x) + c\hat{\lambda}(q', 0^{lg(x)})$.

(e) $\lambda(q + cq', x) = \lambda(q, x) + c\lambda(q', 0^{lg(x)})$.[5]

(f) $\hat{\lambda}(q, x) = \hat{\lambda}(q, 0^{lg(x)}) + \hat{\lambda}(0, x)$.

We shall also need certain facts concerning equivalent states and minimal machines [8].

DEFINITION. Let $M_i = \langle Q_i, \Sigma, \Delta, \delta_i, \lambda_i \rangle$, $i = 1, 2$ be two sequential machines. State q_1 in Q_1 is said to be *equivalent* to state q_2 in Q_2 (written $q_1 \equiv q_2$) if

$$\lambda_1(q_1, x) = \lambda_2(q_2, x)$$

for all x in Σ^*. A sequential machine is said to be *minimal* if it has no distinct equivalent states.

The main result on minimization of LSM's is stated below and is due (independently) to Gill [6] and to Cohn and Even [2].

THEOREM. *For each* LSM M, *one can effectively construct an* LSM M' *such that* M' *is minimal and* $F(M) = F(M')$.

[3] In both [6] and [2], the LSM's are defined over finite fields, but some of the results are true over arbitrary fields and others are true over arbitrary "computable" fields. Computable fields are discussed in Section 4.

[4] $0^{lg(x)}$ denotes the zero-vector of F_k concatenated with itself $lg(x)$ times.

[5] The concatenation is componentwise. For any $c \in F$, $x = a_1 \ldots a_m$, $a_i \in F_l$, we have $cx = (ca_1) \ldots (ca_m)$.

In order to discuss whether or not M' is unique in the previous theorem, the following concepts are required.

DEFINITION. Let $M_i = \langle Q_i, \Sigma, \Delta, \delta_i, \lambda_i \rangle$ be sequential machines for $i = 1, 2$. There is a *homomorphism* φ from M_1 into (onto) M_2 if φ is a map from Q_1 into (onto) Q_2 such that for each $(q, a) \in Q_1 \times \Sigma$, $\varphi\delta_1(q, a) = \delta_2(\varphi q, a)$ and $\lambda_1(q, a) = \lambda_2(\varphi q, a)$. M_1 is *isomorphic* to M_2 if φ is a one–to–one homomorphism from M_1 onto M_2.

In the preceding theorem, M' is unique up to isomorphism. See [*9*] for other properties of homomorphisms and LSM's.

SECTION 1. FINITENESS CONDITIONS ON SEQUENTIAL MACHINES

In this section, we introduce some finiteness conditions on sequential machines. Some of these conditions, particularly the finite memory concept, are intimately related to LSM's. We give the relations between the various properties and characterize LSM's with these properties.

It is hoped that this section will clear up a certain amount of confusion in the literature concerning finite memory and definite automata.

DEFINITIONS

Let $M = \langle Q, \Sigma, \Delta, \delta, \lambda \rangle$ be a sequential machine. With each positive integer p, we associate eight conditions which M may or may not satisfy.

(0) (Feedback free condition) For each $x \in \Sigma^p\Sigma^*$;
$q_1, q_2 \in Q$, $\delta(q_1, x) = \delta(q_2, x)$.

(1) For each $x \in \Sigma^p\Sigma^*$; $q_1, q_2 \in Q$,
$\delta(q_1, x) \equiv \delta(q_2, x)$.

(2) For each $x \in \Sigma^p\Sigma^*$; $q \in Q$, $\delta(q, x) \equiv \delta(q, x^{(p)})$.[6]

(3) For each $x \in \Sigma^p\Sigma^*$; $q \in Q$, $\hat{\lambda}(q, x) = \hat{\lambda}(q, x^{(p)})$.

(4) (Definite condition) For each $x \in \Sigma^p\Sigma^*$ and each $q_1, q_2 \in Q$,
$\hat{\lambda}(q_1, x) = \hat{\lambda}(q_2, x^{(p)})$.

(5) (Finite memory condition) For each $x \in \Sigma^p\Sigma^*$; $q_1, q_2 \in Q$,
$\lambda(q_1, x) = \lambda(q_2, x)$ implies $\delta(q_1, x) \equiv \delta(q_2, x)$.

(6) (p-Diagnosability Condition) For each $x \in \Sigma^p\Sigma^*$; $q_1, q_2 \in Q$,
$\lambda(q_1, x) = \lambda(q_2, x)$ implies $q_1 \equiv q_2$.

(7) For each $x \in \Sigma^p\Sigma^*$; $q_1, q_2 \in Q$,
$\lambda(q_1, x) = \lambda(q_2, x)$ implies $q_1 = q_2$.

[6] For each $x \in \Sigma^p\Sigma^*$, write $x = yx^{(p)}$ where $x^{(p)}$ denotes the last p symbols of x.

Condition (0) is a formal version of the "feedback free" definition. Condition (1) is a weaker version of condition (0), equivalent to it for minimal machines. Conditions (2) and (3) (which are equivalent) are variants of the conditions of definiteness [*8*] but are not equivalent to it.[7] Condition (4) is the definition of definiteness while (5) is the definition of the finite memory property ([*1*], [*6*]). Condition (6) says that all sufficiently long sequences are "diagnosing sequences" [*6*]. Condition (7) is a different definition of diagnosing sequences due to Cohn [*1*].

We now begin to relate the conditions.

PROPOSITION 1.1. *If M is a sequential machine which satisfies condition* (i) $(0 \leqslant i \leqslant 7)$ *for* $p = p_0$, *then* M *satisfies condition* (i) *for all* $p \geqslant p_0$.

The proof is obvious and is omitted. Now we relate conditions (1) and (0).

PROPOSITION 1.2. *Let M be a sequential machine.* (0) *implies* (1) *and if M is minimal,* (1) *implies* (0). *There exists a finite* LSM *which satisfies* (1) *for* $p = 1$, *but does not satisfy* (0) *for any* $p \geqslant 1$.

Proof. Obvious.

We now consider (7).

PROPOSITION 1.3. (7) *implies* (6) *and if M is minimal* (6) *implies* (7). *There exists a finite* LSM *which satisfies* (6) *but does not satisfy* (7).

Proof. Obvious.

Next we relate (1) and (2).

THEOREM 1.1. (1) *implies* (2), *but there exists a finite minimal* LSM *which satisfies* (2) *for* $p = 1$, *but does not satisfy* (1) *for any* $p \geqslant 1$.

Proof. Let $M = \langle Q, \Sigma, \Delta, \delta, \lambda \rangle$ be a sequential machine. Let $x \in \Sigma^p\Sigma^*$ and write $x = yx^{(p)}$. For any $q_1 \in Q$,

$$\delta(q_1, x) = \delta(\delta(q_1, y), x^{(p)}) = \delta(q_2, x^{(p)})$$

where $q_2 = \delta(q_1, y)$. If M satisfies condition (1), $\delta(q_1, x^{(p)}) \equiv \delta(q_2, x^{(p)})$ so that $\delta(q_1, x) \equiv \delta(q_1, x^{(p)})$ and so M satisfies condition (2).

Consider the LSM $M = \langle GF(2), (1), (0), (1), (0) \rangle$. Clearly M satisfies (2) for $p = 1$, but does not satisfy (1) for any $p \geqslant 1$.

Next, we establish the relationship between (2) and (3).

[7] If M is a sequential machine with an initial state which is connected then conditions (2) and (3) are equivalent to conditions (4) and (1).

THEOREM 1.2. *Let* $M = \langle Q, \Sigma, \Delta, \delta, \lambda \rangle$ *be a sequential machine. Condition* (2) *holds for* p [i.e., *for each* $x \in \Sigma^p\Sigma^*$; $q \in Q$, $\delta(q, x) = \delta(q, x^{(p)})$] *if and only if condition* (3) *holds for* $p + 1$ [i.e., *for each* $x \in \Sigma^{p+1}\Sigma^*$, $\hat{\lambda}(q, x) = \hat{\lambda}(q, x^{(p+1)})$].

Proof. Let $q \in Q$ and $x \in \Sigma^{p+1}\Sigma^*$. Write $x = ya$ where $a \in \Sigma$.

$$\hat{\lambda}(q, ya) = \lambda(\delta(q, y), a) = \lambda(\delta(q, y^{(p)}), a)$$

using condition (2). Thus

$$\hat{\lambda}(q, ya) = \hat{\lambda}(q, y^{(p)}a) = \hat{\lambda}(q, x^{(p+1)}).$$

Conversely, suppose that $x \in \Sigma^{p+1}\Sigma^*$ and that $\delta(q, x) \not\equiv \delta(q, x^{(p)})$. Then there exists $z \in \Sigma^+$ such that

$$\hat{\lambda}(q, xz) \neq \hat{\lambda}(q, x^{(p)}z). \qquad (*)$$

However, by condition (3) for $p + 1$.

$$\hat{\lambda}(q, xz) = \hat{\lambda}(q, (xz)^{(p+1)}).$$

Since $z \neq \Lambda$, we note that $(xz)^{(p+1)} = (x^{(p)}z)^{(p+1)}$. Therefore,

$$\hat{\lambda}(q, xz) = \hat{\lambda}(q, (x^{(p)}z)^{(p+1)}) = \hat{\lambda}(q, x^{(p)}z)$$

using condition (3) again. But $\hat{\lambda}(q, xz) = \hat{\lambda}(q, x^{(p)}z)$ contradicts $(*)$ and establishes that $\delta(q, x) \equiv \delta(q, x^{(p)})$.

Next, we relate conditions (5) and (6).

THEOREM 1.3. (6) *implies* (5) *but there exists a finite minimal sequential machine* M *which satisfies* (5) *for* $p = 1$ *but does not satisfy* (6) *for any* $p \geqslant 1$. *Moreover, for* LSM's, (5) *and* (6) *are equivalent.*

Proof. To show that (6) implies (5), assume $\lambda(q_1, x) = \lambda(q_2, x)$ for each $x \in \Sigma^p\Sigma^*$; $q_1, q_2 \in Q$. Then $q_1 \equiv q_2$ by (6). By the "right congruence property" of $\equiv$ (cf [8]), $\delta(q_1, x) \equiv \delta(q_2, x)$.

Let $M = \langle \{q_0, q_1\}, \{0, 1\}, \{0, 1\}, \delta, \lambda \rangle$ where $\delta(q_i, a) = q_1$ for $i, a \in \{0, 1\}$. $\lambda(q_0, a) = a$ and $\lambda(q_1, a) = 1$ for $a \in \{0, 1\}$. M satisfies (5) for $p = 1$, but does not satisfy (6) for any $p \geqslant 1$.

Suppose M is linear and (5) holds. Let $q_1, q_2 \in Q$ and $x \in \Sigma^p\Sigma^*$ and assume that $\lambda(q_1, x) = \lambda(q_2, x)$. Then $\delta(q_1, x) \equiv \delta(q_2, x)$ and for all $y \in \Sigma^*$

$$\lambda(\delta(q_1, x), y) = \lambda(\delta(q_2, x), y).$$

From this and the hypotheses

$$\begin{aligned}\lambda(q_1, xy) &= \lambda(q_1, x)\,\lambda(\delta(q_1, x), y)\\ &= \lambda(q_2, x)\,\lambda(\delta(q_2, x), y) = \lambda(q_2, xy).\\ \lambda(0, xy) + \lambda(q_1, 0^{lg(xy)}) &= \lambda(q_1, xy) = \lambda(q_2, xy)\\ &= \lambda(0, xy) + \lambda(q_2, 0^{lg(xy)}).\end{aligned}$$

Thus $\lambda(q_1, 0^i) = \lambda(q_2, 0^i)$ for all $i \geqslant 0$. [This is true for $i \geqslant lg(x)$ by the above equality and for $i < lg(x)$ by the fact that λ is length preserving]. By using the same identity, ((c) of the first Proposition) $\lambda(q_1, z) = \lambda(q_2, z)$ for all $z \in \Sigma^*$. Thus $q_1 \equiv q_2$.

The relations between (3) and (5) are now derived.

THEOREM 1.4. *There exists a finite minimal sequential machine which satisfies* (3) *for* $p = 1$, *but does not satisfy* (5) *for any* $p \geqslant 1$. *However, if* M *is an* LSM, *then* (3) *implies* (5). *There exists a finite minimal* LSM *which satisfies* (5) *for* $p = 2$, *but does not satisfy* (3) *for any* $p \geqslant 1$.

Proof. Let $M_1 = \langle\{q_1, q_2\}, \{0, 1\}, \{0, 1\}, \delta, \lambda\rangle$ where $\delta(q_i, a) = q_i$ for $i = 1, 2$; $a \in \Sigma$; $\lambda(q_2, 1) = 1$ and $\lambda(q_i, a) = 0$ in all other cases. Clearly, M_1 satisfies (3) for $p = 1$, but does not satisfy (5) for any $p \geqslant 1$.

Let M be an LSM which satisfies (3). We are going to prove that it also satisfies (5).[8] For any $q \in Q$, $x \in \Sigma^+$,

$$\hat{\lambda}(q, x) = \mathbf{C}\mathbf{A}^{lg(x)-1}q + \phi(x)$$

where $\phi(x)$ is a vector depending on x alone and such that if x is a sequence of 0 vectors, $\phi(x) = 0$. (See Proposition, part (b) in the introduction.)

By choosing $x = 0^{t+1}$ i.e., a sequence of $t + 1$ zero vectors we see that for any $t \geqslant p - 1$

$$\mathbf{C}\mathbf{A}^t = \mathbf{C}\mathbf{A}^{p-1}$$

since M satisfies (3).

Now let $x \in \Sigma^p\Sigma^*$; $q_1, q_2 \in Q$ and $\lambda(q_1, x) = \lambda(q_2, x)$. Then

$$\begin{aligned}\mathbf{C}\mathbf{A}^{p-1}q_1 + \phi(x) &= \mathbf{C}\mathbf{A}^{lg(x)-1}q_1 + \phi(x)\\ &= \hat{\lambda}(q_1, x)\\ &= \hat{\lambda}(q_2, x)\\ &= \mathbf{C}\mathbf{A}^{lg(x)-1}q_2 + \phi(x)\\ &= \mathbf{C}\mathbf{A}^{p-1}q_2 + \phi(x),\end{aligned}$$

and so $\mathbf{C}\mathbf{A}^{p-1}q_1 = \mathbf{C}\mathbf{A}^{p-1}q_2$.

[8] It is well known that every LSM satisfies (5) for some p([1], [6]). Here we prove that if an LSM satisfies (3) for a given p, it will satisfy (5) for the same p.

Let $y \in \Sigma^+$. Then

$$\begin{aligned}\hat{\lambda}(\delta(q_1, x), y) &= \hat{\lambda}(q_1, xy) \\ &= \mathbf{C}\mathbf{A}^{lg(xy)-1} q_1 + \phi(xy) \\ &= \mathbf{C}\mathbf{A}^{p-1} q_1 + \phi(xy) \\ &= \mathbf{C}\mathbf{A}^{p-1} q_2 + \phi(xy) \\ &= \mathbf{C}\mathbf{A}^{lg(xy)-1} q_2 + \phi(xy) \\ &= \hat{\lambda}(q_2, xy) \\ &= \hat{\lambda}(\delta(q_2, x), y).\end{aligned}$$

Hence $\delta(q_1, x) \equiv \delta(q_2, x)$ and M satisfies (5).

To complete the proof, consider $M_2 = \langle GF(2), \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \rangle$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\mathbf{C} = (0 \quad 1), \qquad \mathbf{D} = (0).$$

M_2 satisfies (5) for $p = 2$, but does not satisfy (3) for any $p \geqslant 1$ as can be seen by checking $q = \binom{1}{0}$.

Next we relate conditions (1) and (4).

THEOREM 1.5. *A sequential machine M satisfies* (1) *for p if and only if it satisfies* (4) *for $p + 1$.*

Proof. Suppose that condition (1) is satisfied for p. Let $q_1, q_2 \in Q$, $x \in \Sigma^*$, $z \in \Sigma^p$, $a \in \Sigma$. Then

$$\hat{\lambda}(q_1, xza) = \hat{\lambda}(\delta(q_1, x), za) = \hat{\lambda}(\delta(\delta(q_1, x), z), a).$$

By (1) $\delta(\delta(q_1, x), z) \equiv \delta(q_2, z)$ so that

$$\hat{\lambda}(q_1, xza) = \hat{\lambda}(\delta(q_2, z), a) = \hat{\lambda}(q_2, za) = \hat{\lambda}(q_2, (xza)^{(p+1)})$$

because $(xza)^{(p+1)} = za$. So M satisfies (4) for $p + 1$.

Suppose M satisfies (4) for $p + 1$. Let $q_1, q_2 \in Q$, $x \in \Sigma^p \Sigma^*$, $z \in \Sigma^+$. Then

$$\hat{\lambda}(q_1, xz) = \hat{\lambda}(q_2, (xz)^{(p+1)}) = \hat{\lambda}(q_2, xz).$$

Therefore $\delta(q_1, x) \equiv \delta(q_2, x)$ and so M satisfies (1) for p.

In the following theorem, we describe the implications between (4) and (5).

THEOREM 1.6. *If a sequential machine satisfies* (4) *for $p + 1$, it satisfies* (5) *for p. However, there exists a finite minimal* LSM *which satisfies* (5) *for $p = 1$, but does not satisfy* (4) *for any $p \geqslant 1$.*

Proof. If a sequential machine M satisfies (4) for $p + 1$, then it satisfies (1) for p (Theorem 1.5). But then M obviously satisfies (5) for p as well.

The LSM $\langle GF(2), (1), (0), (1), (0)\rangle$ is such that it satisfies (5) for $p = 1$, but does not satisfy (4) for any $p \geqslant 1$.

In order to complete our picture concerning the relations between the eight conditions, we need only consider whether (0) implies (7).

THEOREM 1.7. *There exists a finite minimal sequential machine M which satisfies* (0) *for $p = 1$, but does not satisfy* (7) *for any $p \geqslant 1$. There exists a finite* LSM *M' which satisfies* (0) *for $p = 1$, but does not satisfy* (7) *for any $p \geqslant 1$. However, if a minimal* LSM *satisfies* (0), *it also satisfies* (7).

Proof. $M = \langle\{q_0, q_1\}, \{0, 1\}, \{0, 1\}, \delta, \lambda\rangle$ where $\delta(q_i, a) = q_1$ for $i, a \in \{0, 1\}$ and $\lambda(q_0, a) = a$, $\lambda(q_1, a) = 0$ for $a \in \{0, 1\}$ satisfies (0) for $p = 1$ but does not satisfy (7) for any $p \geqslant 1$.

$M' = \langle GF(2), (0), (0), (0), (0)\rangle$ satisfies (0) for $p = 1$ but does not satisfy (7) for any $p \geqslant 1$.

If a minimal LSM satisfies (0) for p, it satisfies (1) for p (Proposition 1.2). But then it satisfies (4) for $p + 1$ (Theorem 1.5), and (5) for p (Theorem 1.6). Therefore it satisfies (6) for p (Theorem 1.3) and since it is minimal, (7) for p as well (Proposition 1.3).

We can also verify the following simple proposition whose straight forward proof is omitted.

PROPOSITION 1.4. *If a sequential machine M satisfies* (7) *then M is minimal.*

Our results can be summarized by the following two tables. If at the ith row and jth column (start the indexing at 0 and take $p \dot{-} 1 = \max\{p - 1, 1\}$), the table has a

- $\binom{Y}{r}$ then if a (linear) sequential machine satisfies (i) for p then it satisfies (j) for r.
- N then there is a finite minimal (linear) sequential machine which satisfies (i) for some p, but does not satisfy (j) for any $p \geqslant 1$.
- $\binom{Y/N}{p}$ then if a minimal (linear) sequential machine satisfies (i) for p, then it satisfies (j) for p, but there is a finite (linear) sequential machine which satisfies (i) for some p but does not satisfy (j) for any $p \geqslant 1$.

As an example of how these tables are obtained from the theorems, we give the reasoning behind the N in the 7th row and the 0th column of Table II (for LSM's).

Suppose we were wrong in placing an N there. Then every finite minimal LSM which satisfies (7) for some p will satisfy (0) for some $q \geqslant 1$. Let M be the finite minimal LSM which satisfies (5) for $p = 2$ but does not satisfy (3) for any $p \geqslant 1$

TABLE I

RELATIONS BETWEEN THE CONDITIONS FOR SEQUENTIAL MACHINES

	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(0)	Y p	Y p	Y p	Y p	Y p	Y p	N	N
(1)	Y/N p	Y p	Y p	Y $p+1$	Y $p+1$	Y p	N	N
(2)	N	N	Y p	Y $p+1$	N	N	N	N
(3)	N	N	Y $p-1$	Y p	N	N	N	N
(4)	Y/N p	Y $p-1$	Y $p-1$	Y p	Y p	Y $p-1$	N	N
(5)	N	N	N	N	N	Y p	N	N
(6)	N	N	N	N	N	Y p	Y p	Y/N p
(7)	N	N	N	N	N	Y p	Y p	Y p

(Theorem 1.4.) M satisfies (6) for $p = 2$ (Theorem 1.3.) and since it is minimal, M satisfies (7) for $p = 2$ (Proposition 1.3). By the assumption above M satisfies (0) for some $q \geqslant 1$. But then M satisfies (1) for q (Proposition 1.2), M satisfies (2) for q (Theorem 1.1.) and M satisfies (3) for $q + 1$ (Theorem 1.2.). This contradicts the choice of M and so proves that we were right in putting N in the 7th row and 0th column of Table II.

There is a convenient pictorial summary of the tables which can be obtained as directed graphs. A directed path from node i to node j corresponds to a Y in the (i, j) entry of the table. No path corresponds to a N or a Y/N. Equivalent conditions are encircled.

These graphs clearly indicate the relative strength of conditions (0)–(7).

Next, we turn to the conditions under which linear machines have these properties.

THEOREM 1.8. *Let* $M = \langle F, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \rangle$ *be an* LSM. *M is feedback free* [i.e., *satisfies condition* (0) *for p, where p is minimal*] *if and only if* $\mathbf{A}$ *is nilpotent*[9] *of degree p.*

[9] A matrix $\mathbf{A}$ is said to be *nilpotent of degree p* if p is the least positive integer for which $\mathbf{A}^p = \mathbf{0}$.

TABLE II

RELATIONS BETWEEN THE CONDITIONS FOR LSM'S

	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(0)	Y p	Y p	Y p	Y p	Y p	Y p	Y p	Y/N p
(1)	Y/N p	Y p	Y p	Y $p+1$	Y $p+1$	Y p	Y p	Y/N p
(2)	N	N	Y p	Y $p+1$	N	Y p	Y p	Y/N p
(3)	N	N	Y $p-1$	Y p	N	Y p	Y p	Y/N p
(4)	Y/N p	Y $p-1$	Y $p-1$	Y p	Y p	Y $p-1$	Y p	Y/N p
(5)	N	N	N	N	N	Y p	Y p	Y/N p
(6)	N	N	N	N	N	Y p	Y p	Y/N p
(7)	N	N	N	N	N	Y p	Y p	Y p

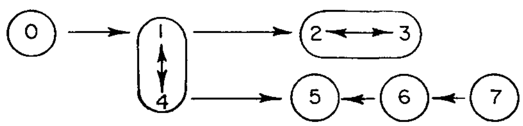

FIG. 1. Graph for Sequential Machines.

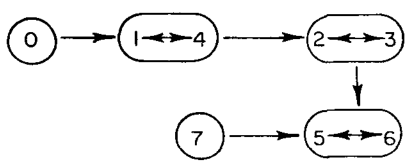

FIG. 2. Graph for LSM's.